Mechanical Vibration

Iva Petríková

Department of Applied Mechanics

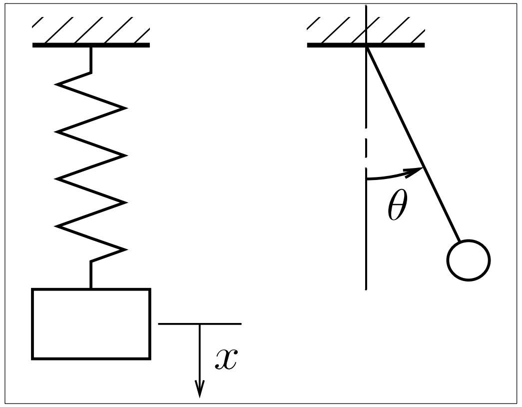
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Introduction, basic definitions

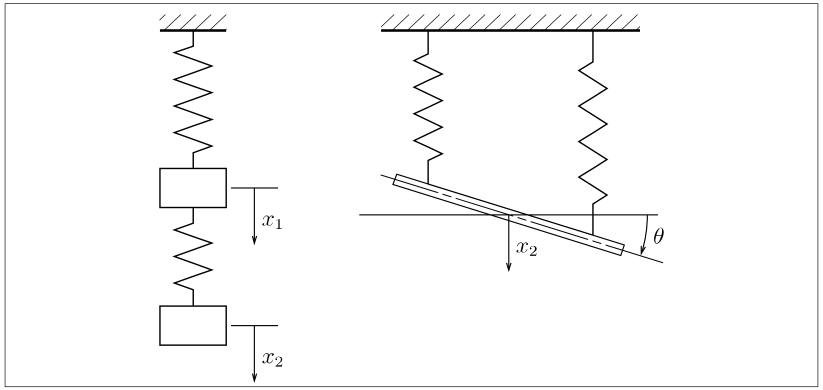
- Oscillatory process – alternate increases or decreases of physical quantities (displacements, velocities, accelerations)
- Oscillatory motion is periodic motion
- Vibration of turbine blades, vibration of machine tools, electrical oscillation, sound waves, vibration of engines, torsional vibration of shafts, vibration of automobiles etc.
- Mechanical vibration - mechanisms and machines, buildings, bridges, vehicles, aircrafts – cause mechanical failure
- Harmonic, periodic general motion

Elastic elements

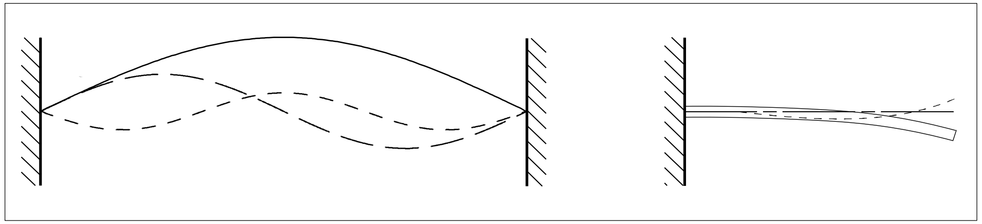
- Discrete elements (masses) + linear and torsional springs
- Continuos structural elements – beams and plates
- Number of degrees of freedom (DOF) – minimum number of coordinates



Single degree of freedom systems



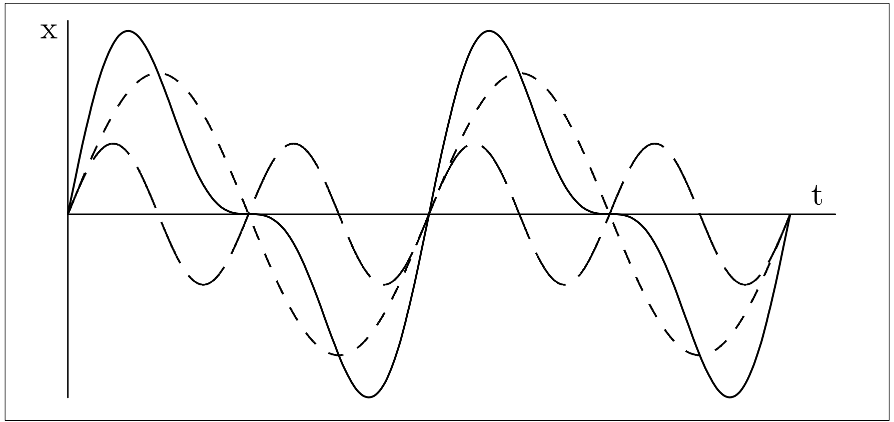
Two degree of freedom systems



System with infinite number DOF

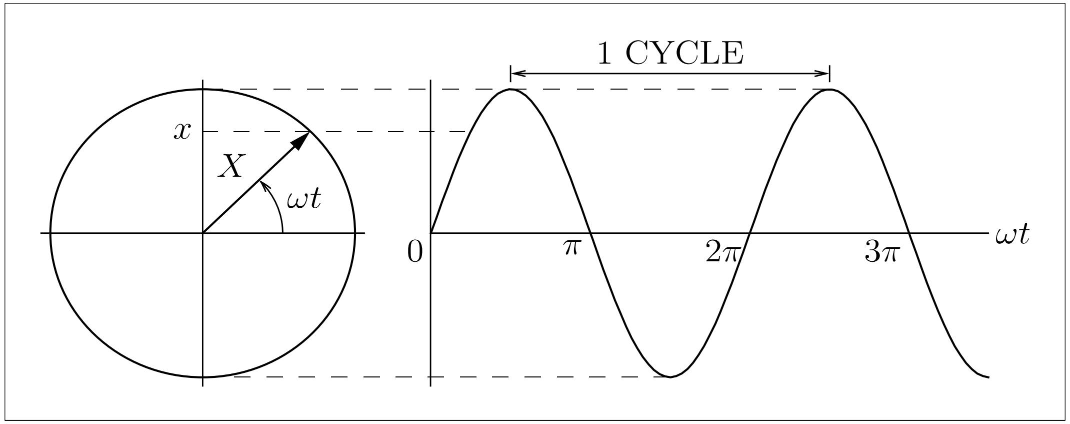
Oscillatory motions

- Periodic motion with harmonic components



Periodic motion repeating itself after a certain time interval.

- Harmonic motion



The simplest form of periodic motion is harmonic motion – sin, cos

Harmonic motion

- Displacement of harmonic motion is given:

$$x = X \sin(\omega t + \varphi)$$

- x, x(t) ... displacement [m]
- X ... amplitude of displacement [m]
- $(\omega t + \varphi)$ ... phase
- ω ... angular velocity [s^{-1}]
- T ... natural period of oscillation [s]
- f ... frequency [s^{-1}], [Hz] - Hertz
- φ ... phase angle

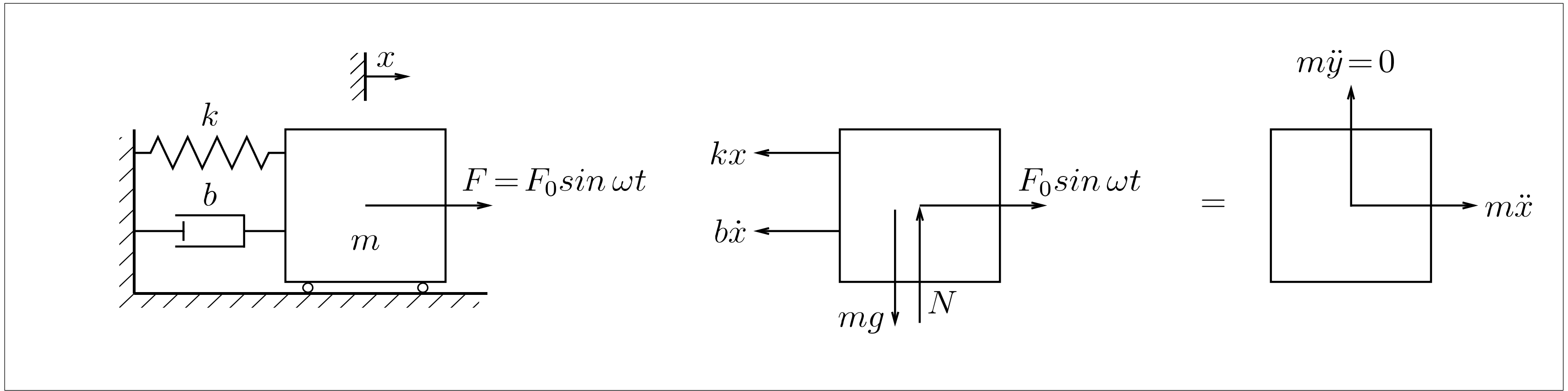
$$T = \frac{2\pi}{\omega}$$

$$\omega = 2\pi f$$

- velocity: $$\dot{x} = \frac{dx}{dt} = \omega X \cos(\omega t + \varphi)$$
- ωX ... amplitude of velocity [ms^{-1}]
- acceleration: $$\ddot{x} = \frac{d^2 x}{dt^2} = -\omega^2 X \sin(\omega t + \varphi)$$
- $-\omega^2 X$ ... amplitude of acceleration [ms^{-2}]

Simple degree of freedom systems

mass, spring, damper, harmonic excitation



Forcing function – harmonic excitation

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Damped free vibration

$$m\ddot{x} + b\dot{x} + kx = 0$$

Undamped free vibration

$$m\ddot{x} + kx = 0$$

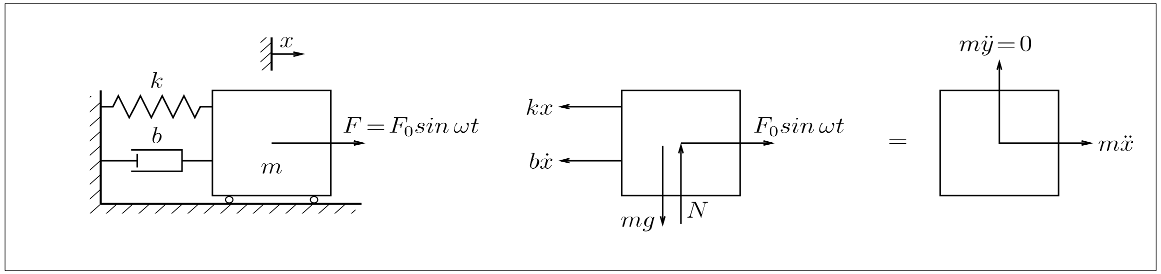
evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
TECHNICKÁ UNIVERZITA V LIBERCI
www.tul.cz
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Simple degree of freedom systems

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$F(t) = F_0 \sin \omega t$$

m ... mass
b ... (viscous) damping coefficient
k ... stiffness coefficient
F_0 ... amlitudes of force
ω ... frequency of harmonic force
b_{cr} ... critical damping coefficient
ζ ... damping factor



$\Omega = \sqrt{\frac{k}{m}}$ ... natural (circular) frequency

$\eta = \frac{\omega}{\Omega}$ ... frequency ratio

Undamped free vibration

$$m\ddot{x} + kx = 0$$

Solution of the 2nd order differential equation

Assumed solution $x(t) = Ae^{\lambda t}$

Characteristic equation:

$$m\lambda^2 + k = 0$$

$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda^2 = -\frac{k}{m}$$

$$\lambda_{1,2} = \pm i\sqrt{\frac{k}{m}}$$

$$\lambda_{1,2} = \pm i\Omega$$

$$\Omega = \sqrt{\frac{k}{m}}$$ Natural frequency of the single DOF systems

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \ddot{x} + \Omega^2 x = 0$$

$$x(t) = Ae^{i\Omega t} + Be^{-i\Omega t}$$ free vibration

Two arbitrary constants A a B, determined from initial conditions:

$$x(0) = x_0, \dot{x}(0) = v_0$$

$$\dot{x}(t) = i\Omega\left(Ae^{i\Omega t} - Be^{-i\Omega t}\right)$$

$$x_0 = A + B$$

$$v_0 = i\Omega(A - B)$$

$$A = \frac{1}{2}\left(\frac{ix_0\Omega + v_0}{i\Omega}\right)$$

$$B = \frac{1}{2}\left(\frac{ix_0\Omega - v_0}{i\Omega}\right)$$

Damped free vibration

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solution of linear differential equation of 2nd order:

$$m\lambda^2 + b\lambda + k = 0$$

$$\lambda_{1,2} = \frac{-b}{2m} \pm \sqrt{b^2 - 4mk}\,\frac{1}{2m}$$

$$\lambda_{1,2} = \frac{-b}{2m} \pm i\sqrt{\frac{4mk}{4m^2} - \left(\frac{b}{2m}\right)^2} = \frac{-b}{2m} \pm i\sqrt{\frac{k}{m}}\sqrt{1 - \left(\frac{b}{2m\sqrt{\frac{k}{m}}}\right)^2}$$

$$\lambda_{1,2} = \frac{-b}{2m} \pm i\Omega\sqrt{1 - \left(\frac{b}{2\sqrt{km}}\right)^2} = \frac{-b}{2m} \pm i\Omega\sqrt{1 - \zeta^2}$$

$$\frac{b}{2\sqrt{km}} = \frac{b}{b_{CR}} = \zeta$$ damping factor

$$b_{CR} = 2\sqrt{km}$$ critical damping coefficient

$$\Omega\sqrt{1 - \zeta^2} = \Omega_D$$ damped natural frequency

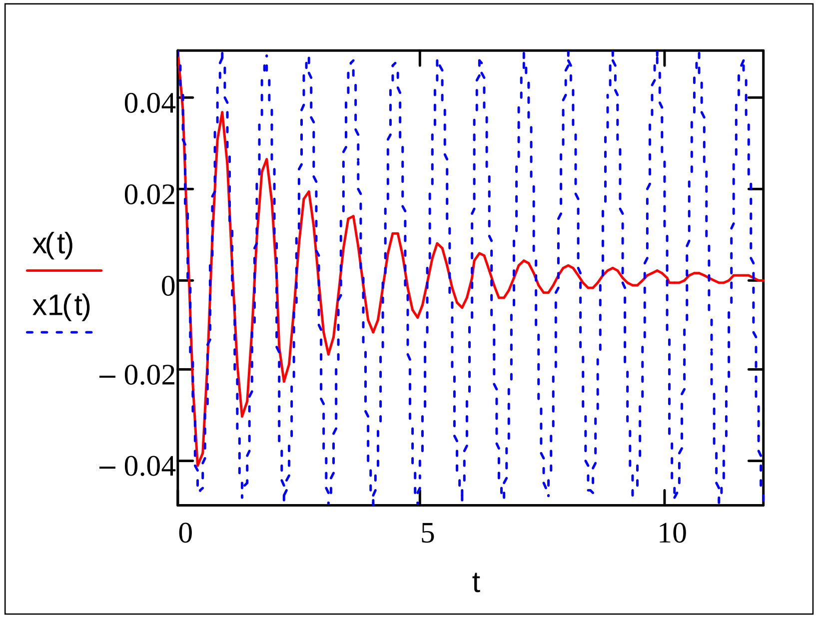
$$\lambda_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\Omega$$

$$x(t) = Ae^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\Omega t} + Be^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\Omega t}$$

$\zeta > 1$ overdamped system

$\zeta < 1$ underdamped system

$\zeta = 1$ critically damped system



Undamped --- and damped free vibration ---

Damped free vibration

Overdamped system: displacement becomes the sum of two decaying exponentials with initial value of A+B, no vibration takes place, the body tends to creep back to the equilibrium position – APERIODIC MOTION (Fig.1)

$\zeta > 1$

$$x = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\Omega t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\Omega t}$$

Underdamped system: displacement is oscillatory with diminishing amplitude (Fig.2). Frequency of oscillation is less than that of the undamped case by the factor $\sqrt{1-\zeta^2}$.

$\zeta < 1$

$$x = e^{-\zeta\Omega t}[C_1 e^{i\sqrt{1-\zeta^2}\Omega t} + C_2 e^{-i\sqrt{1-\zeta^2}\Omega t}] =$$
$$= e^{-\zeta\Omega t}\left(A\cos\Omega_D t + B\sin\Omega_D t\right) =$$
$$= Ce^{-\zeta\Omega t}\sin\left(\sqrt{1-\zeta^2}\Omega t + \gamma\right)$$

Critical damping:

$\zeta = 1$

$$x = [A + Bt]e^{-\Omega t}$$

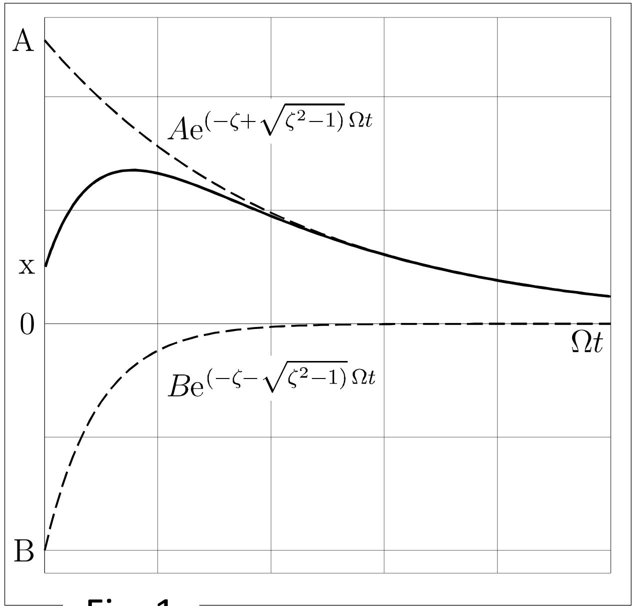


Fig. 1

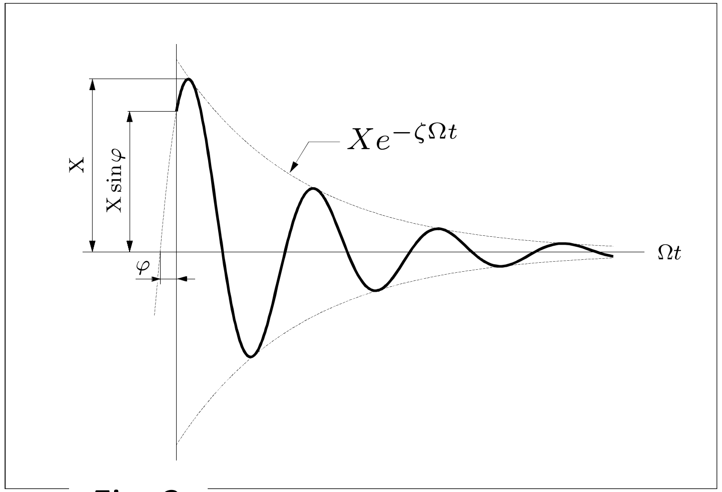


Fig. 2

Damped free vibration – Logarithmic decrement

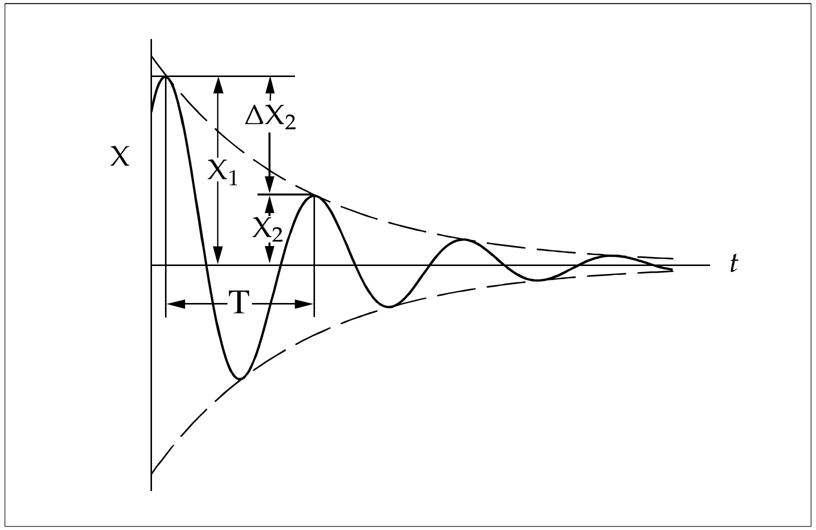
Natural logarithm of the ratio of any two amplitudes

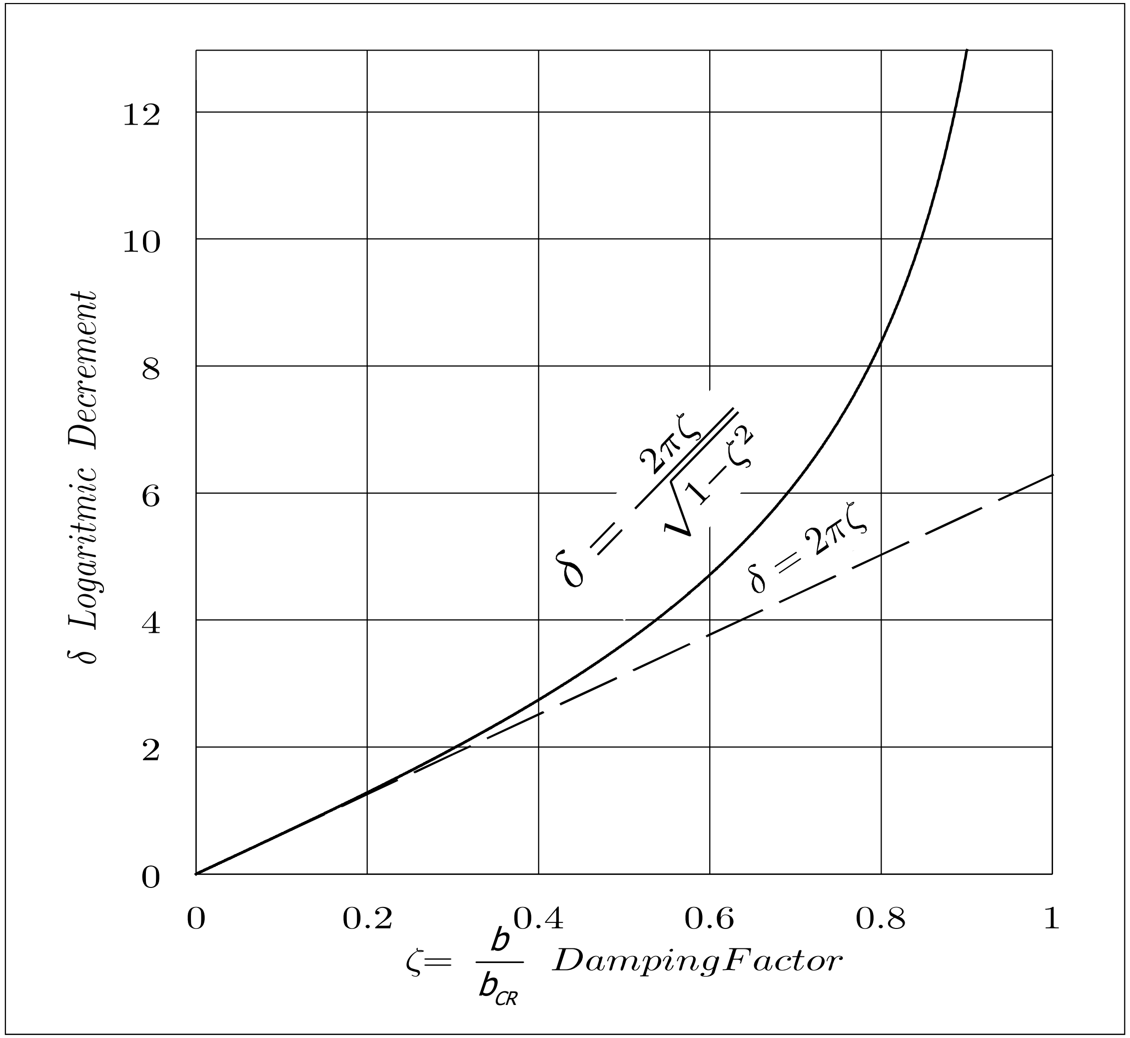
$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{x(t)}{x(t+T)} = \ln\frac{e^{-\zeta\Omega t}\left(A\cos\Omega_D t + B\sin\Omega_D t\right)}{e^{-\zeta\Omega(t+T)}\left(A\cos\Omega_D t + B\sin\Omega_D t\right)} =$$

$$= \ln\frac{1}{e^{-\zeta\Omega T}} = \zeta\Omega T = \zeta\Omega\frac{2\pi}{\Omega_T} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$\delta \doteq 2\pi\zeta$ for $b << b_{CR}$

$$\delta = \ln\frac{x(t)}{x(t+nT)} \doteq 2\pi n\zeta \Rightarrow damping\ \ factor\ \zeta = \frac{\delta}{2\pi n}$$





Forced vibration, harmonic excitation

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin \omega t$$

1. Forced undamped vibration
   - homogenous solution of equation $m\ddot{x} + kx = 0$
   - particular solution of equation $m\ddot{x} + kx = F_0 \sin \omega t$

Solution of dif. equation with right side → steady state oscillation (response)

Assumed solution in the form of harmonic function:

$$x_p = a_1 \sin \omega t + a_2 \cos \omega t$$

$$\ddot{x}_p = -\omega^2 x_p = -\omega^2 \left(a_1 \sin \omega t + a_2 \cos \omega t\right)$$

dif. equation: $\left(k - m\omega^2\right) a_1 \sin \omega t + \left(k - m\omega^2\right) a_2 \cos \omega t = F_0 \sin \omega t$

Comparing of coeficients at function sin a cos on the both side of the equation → amplitude a_1

$$a_1 = \frac{F_0}{k - m\omega^2}, \; a_2 = 0$$

Forced vibration, harmonic excitation

Particular solution: $x_p = \frac{F_0}{k - m\omega^2} \sin \omega t$

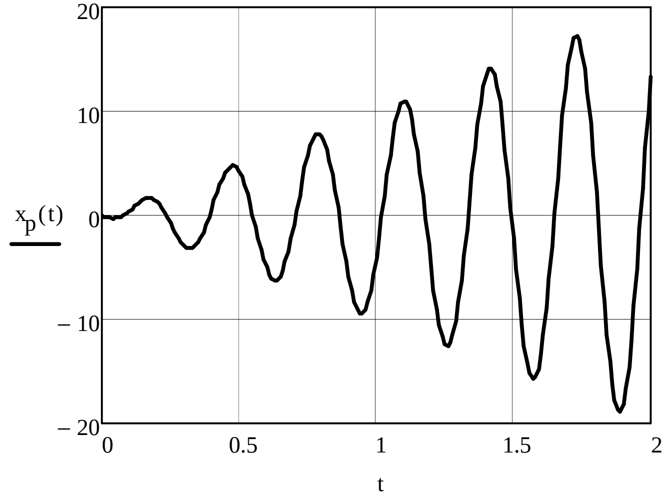
$$a = \frac{F_0}{k - m\omega^2} = \frac{F_0}{k}\frac{1}{1 - \frac{m}{k}\omega^2} = a_{ST}\frac{1}{1-\eta^2}$$

$$a_{ST} = \frac{F_0}{k}$$

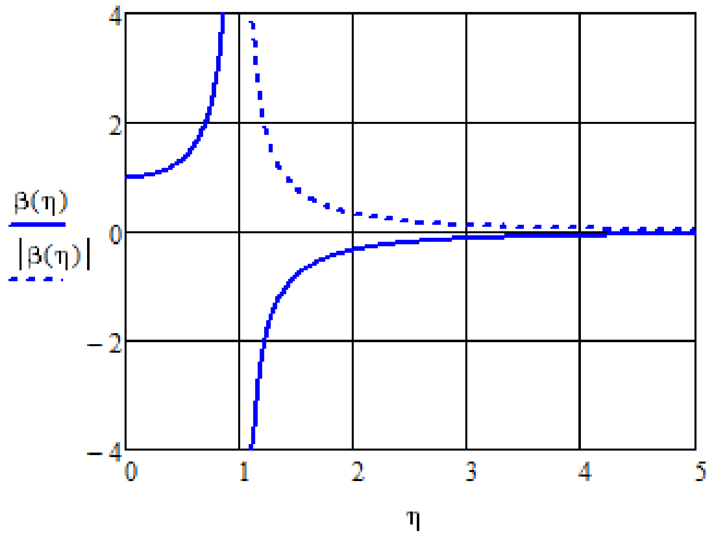
$$\frac{a}{a_{ST}} = \frac{1}{1-\eta^2}$$

resonance – frequency of exciting force equels to natural frquency of the system

$$\eta = 1 \Rightarrow \omega = \Omega \Rightarrow \frac{a}{a_{ST}} \to \infty$$



Vibration of the single DOF system in resonance.



Resonance curve – undamped system

Forced vibration

$$m\ddot{x}+b\dot{x}+kx=F(t)$$

$$m\ddot{x}+b\dot{x}+kx=F_0\sin\omega t \qquad (1)$$

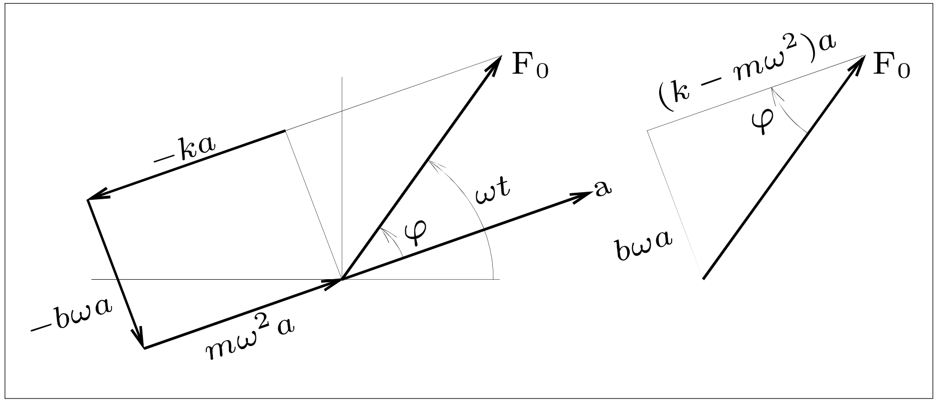
harmonic force

1. $m\ddot{x}+b\dot{x}+kx=0$ — homogenous solution

2. $m\ddot{x}+b\dot{x}+kx=F_0\sin\omega t$ → steady-state solution (response)

$x=a\sin(\omega t-\varphi)$ — assumed solution of differential equation (1)

$$-ma\omega^2\sin(\omega t-\varphi)+ba\omega\cos(\omega t-\varphi)+ka\sin(\omega t-\varphi)=F_0\sin\omega t$$



Vectors' diagram

$$a=\frac{F_0}{\sqrt{\left(k-m\omega^2\right)^2+(b\omega)^2}}=\frac{F_0}{k}\frac{1}{\sqrt{\left(1-\frac{m}{k}\omega^2\right)^2+\left(\frac{b}{k}\omega\right)^2}}$$

$$\tan\varphi=\frac{b\omega}{k-m\omega^2}$$

Single degreee of freedom system

Free damped vibration	Free undamped vibration	Forced damped vibration
$m\ddot{x}+b\dot{x}+kx=0$	$m\ddot{x}+kx=0$	$m\ddot{x}+b\dot{x}+kx=F(t)$
Homogenous solution $x_h=e^{-\zeta\Omega t}[Ae^{i\sqrt{1-\zeta^2}\Omega t}+Be^{-i\sqrt{1-\zeta^2}\Omega t}]=$ $=e^{-\zeta\Omega t}(A\cos\Omega_D t+B\sin\Omega_D t)=$ $=Ce^{-\zeta\Omega t}\sin\left(\sqrt{1-\zeta^2}\Omega t+\gamma\right)$	Homogenous solution $x_h=C_1e^{i\Omega t}+C_2e^{-i\Omega t}$ $x_h=A\cos\Omega t+B\sin\Omega t$ $x_h=C\sin(\Omega t+\gamma)$	$m\ddot{x}+b\dot{x}+kx=F_0\sin\omega t$ $x=x_h+x_p$ $x_p=a\sin(\omega t-\varphi)$
Initial conditions $x(t_0)=x_0 \quad \dot{x}(t_0)=v_0$	Initial conditions $x(t_0)=x_0 \quad \dot{x}(t_0)=v_0$	Amplitude of steady state oscillation, steady state response
		$x(t)=e^{-\zeta\Omega t}[Ae^{i\sqrt{1-\zeta^2}\Omega t}+Be^{-i\sqrt{1-\zeta^2}\Omega t}]$ $+\dfrac{F_0\sin(\omega t-\varphi)}{\sqrt{(k-m\omega^2)^2+(b\omega)^2}}$

Forced vibration - magnification factor and phase angle

Solution of equation (1): $x(t) = x_h + x_p$

$$x(t) = Ce^{-\zeta\Omega t}\sin\left(\sqrt{1-\zeta^2}\,\Omega t + \gamma\right) + \frac{F_0 \sin(\omega t - \varphi)}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}$$

Values C a γ are derived from initial conditions.
Amplitudes of steady-state oscillation:

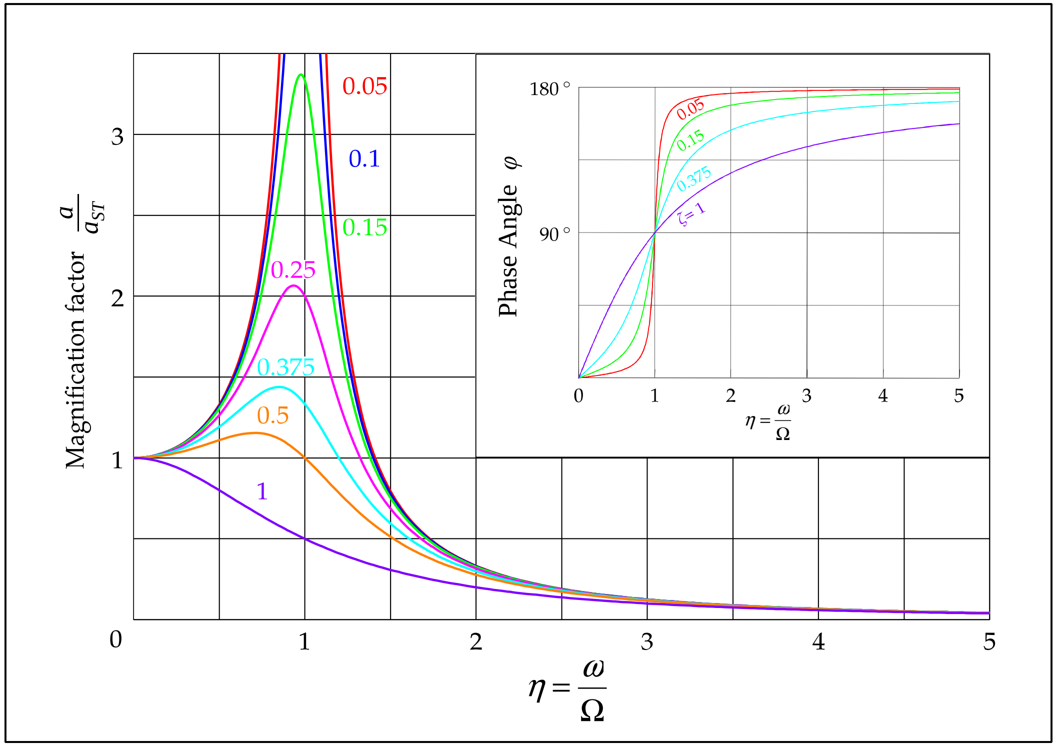
$$a = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} = \frac{F_0}{k}\frac{1}{\sqrt{\left(1 - \frac{m\omega^2}{k}\right)^2 + \left(\frac{b}{k}\omega\right)^2}}$$

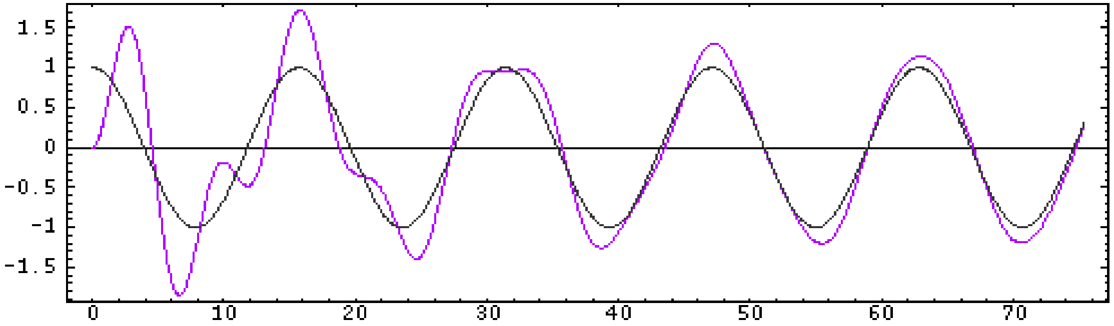
$\frac{F_0}{k} = a_{ST}$ statical deflection of the spring mass system under the action of steady force F_0

$\eta = \frac{\omega}{\Omega}$ frequency ratio

$$\frac{a}{a_{ST}} = \frac{1}{\sqrt{(1-\eta^2)^2 + (2\zeta\eta)^2}}$$ magnification factor

$$\varphi = \arctan\frac{2\xi\eta}{1-\eta^2}$$ phase angle





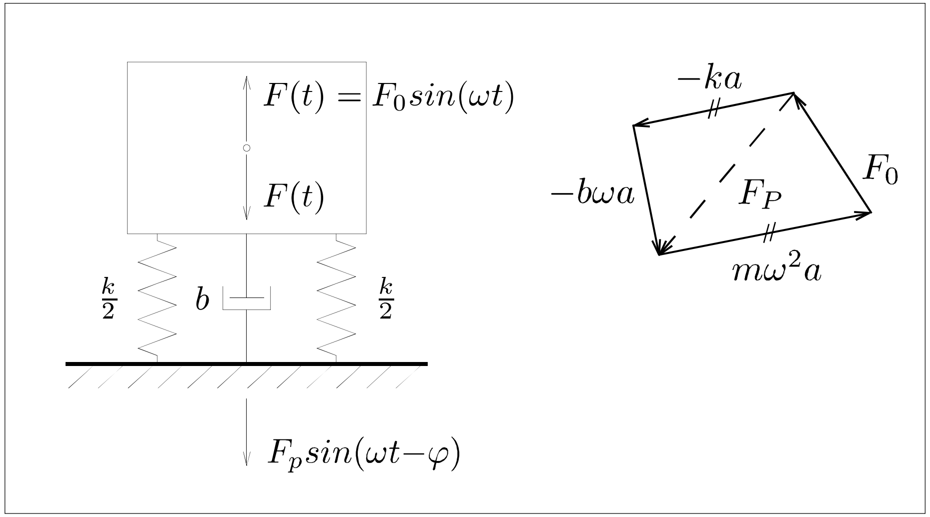
Transient motion, under resonance

Vibration isolation and transmissibility

Machine or engines rigidly attached to a supporting structure, vibration is transmitted directly to the support (often undiserable vibration). Disturbing source must be isolated. Force is trasmitted through springs and damper.

a) ω = const. → η = const. ω >> Ω, η >> 1, small damping factor ζ

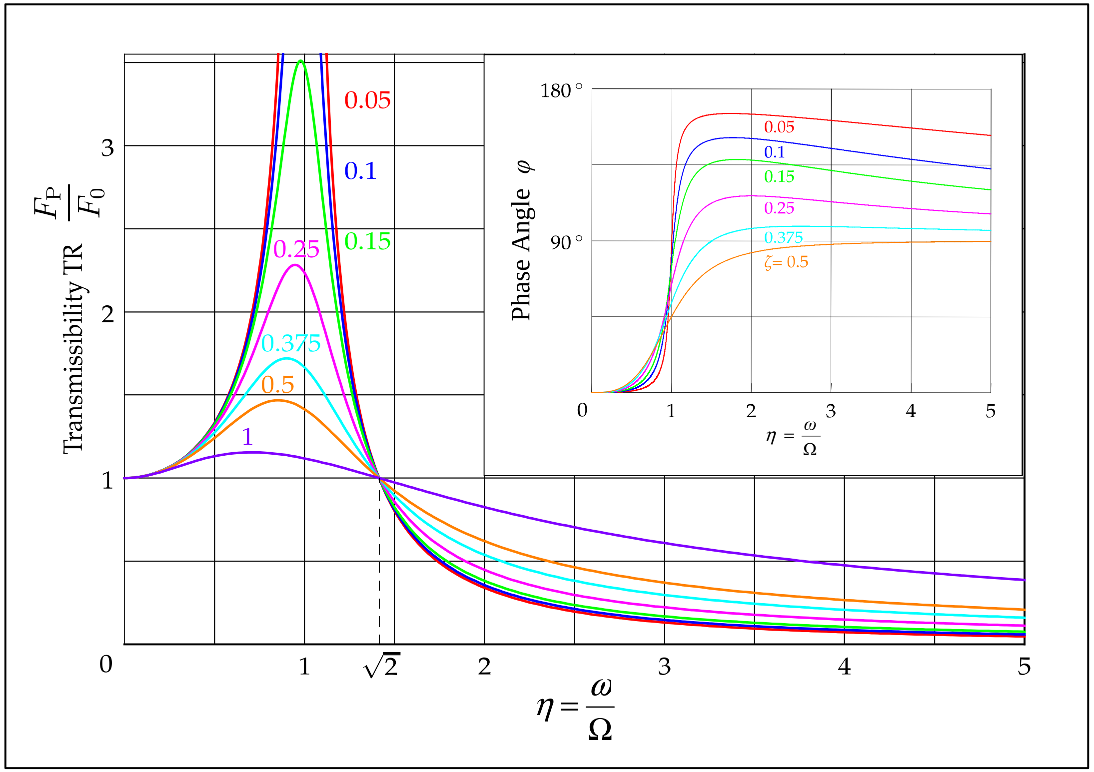
b) ω ≠ const. → isolation, support with damping



$$F_P = \sqrt{(-ka)^2 + (b\omega a)^2} = a\sqrt{k^2 + (b\omega)^2}$$

$$= F_0\sqrt{\frac{k^2+(b\omega)^2}{(k-m\omega^2)^2+(b\omega)^2}}$$

$$\frac{F_P}{F_0} = \sqrt{\frac{1+(2\xi\eta)^2}{(1-\eta^2)^2+(2\xi\eta)^2}}$$

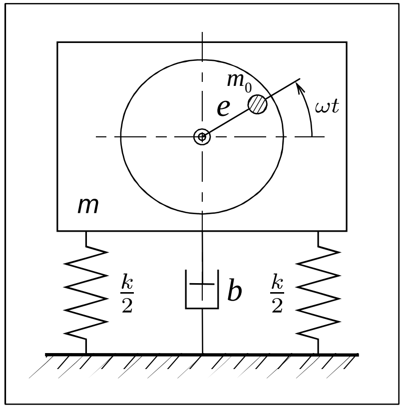


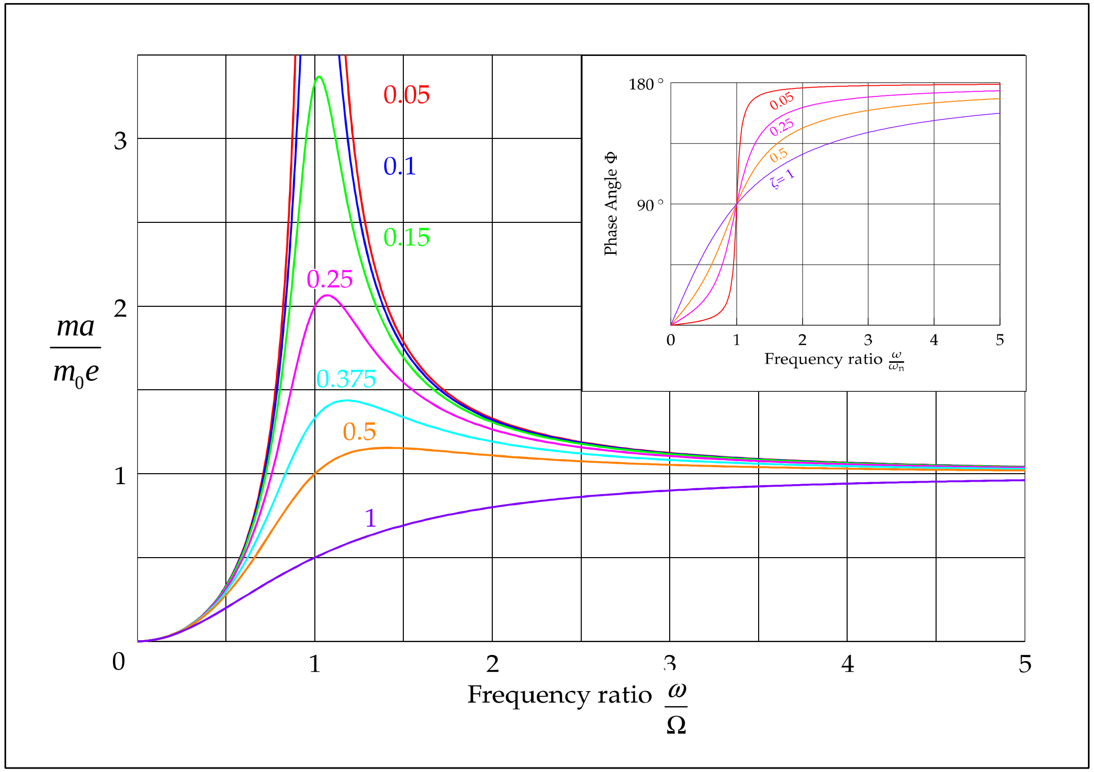
Rotating unbalance

Rotating unbalance systems (gears, wheels, shafts disks which are not perfectly uniform, produce unbalance force which cause excessive vibrations.

m_0 ... unbalance mass

e ... eccentricity





$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$m\ddot{x} + b\dot{x} + kx = m_0 e\omega^2 \sin \omega t$$

$$\frac{a}{m/_{m_0} e} = \frac{\eta^2}{\sqrt{\left(1-\eta^2\right)^2 + \left(2\zeta\eta\right)^2}} \qquad \tan\varphi = \frac{b\omega}{\sqrt{k - m\omega^2}} = \frac{2\xi\eta}{1-\eta^2}$$

Base motion, relative motion

Forced vibration of mechanical systems can be caused by the support motion (vehicles, aircrafts and ships)

harmonic motion $y = y_0 e^{i\omega t}$

$$m\ddot{x} + b(\dot{x} - \dot{y}) + k(x - y) = 0$$

$$m\ddot{x}_r + b\dot{x}_r + kx_r = -m\ddot{y}$$

$$m\ddot{x}_r + b\dot{x}_r + kx_r = m\omega^2 y_0 e^{i\omega t}$$

solution

$$x_p = a_r e^{i(\omega t - \varphi)}$$

$$a_r = \frac{m\omega^2 y_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}$$

$$\frac{a_r}{y_0} = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + (2\zeta\eta)^2}} \qquad \tan\varphi = \frac{2\xi\eta}{1-\eta^2}$$